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Matrix Representation of Quaternions
- Private Edition

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Thanks: DeepSeek
Microsoft, Copilot

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<Introduction>

This is an entry for a concept included in a writing archive called "Three-Dimensional Essays" (available on the Vector website) that I wrote about 15 years ago. However, the archive was mostly devoted to explaining three-dimensional octahedral matrices (48 elements. See something like 48 moves or 48 idols. Buddhist auspicious number.), and this concept was only briefly touched upon. (The attempted proof was about 30 years ago.) This time, I have extracted that section here and provided an explanation. Of course, in the rigorous theories of my predecessors, the concept is given a rather intimidating name (apparently a Lie group), but since I am inferring based on a simple process, I will prioritize using the temporary, convenient name that was used in that process. Since I came up with the name myself, I am not particularly hoping that it will become established, but since it is a name that is paired with the concept, it may be fun to read. Of course, that is assuming you can grasp the concept explained.

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ASCII art (pseudo-diagrams made of characters) is used to explain the charts. Display via a browser will appear as intended, but if you are using a proportional font (a font with slightly different character widths) in a word processor or editor, the charts will be distorted. Please switch to a fixed-pitch font or view the PDF version.

<A simple explanation of quaternions>

The quaternions discovered by Hamilton are a type of imaginary triplet, and they form a group under multiplication, but the commutative law does not hold when they are multiplied. However, if the sign is reversed, the law does hold.

-The product of triplet units has a rotational conceptual parity, and reversing the rotation reverses the sign of the value. (Conceptually homologous to the line above) In other words, (where ^ indicates factorial)

```
 i^2 = j^2 = k^2 = -1

 i*j = -j*i = -k
 j*k = -k*j = -i
 k*i = -i*k = -j

 i*j*k = -1
 k*j*i = +1
```

holds.

<The matrix that started it all>

By chance, I discovered that when the matrix

```
 (0 -1)
 (1  0)
```

is squared, it becomes the negative of the identity element, (-1 0) (0 -1) = -E. (The details of how this came about are at the end of the document.) By regarding this as congruent with the imaginary unit i, I attempted to ambitiously define a matrix equivalent to a quaternion by combining matrix operations, and this document is a record of my trial and error. To state the conclusion first, this could not be achieved with a 2*2 square matrix, so a 4*4 matrix was needed as an extended representation of the definition, and I was able to achieve the desired result.

<The history of trial and error>

Naturally, I had no clues, so I wondered if I could define a quaternion with a 2*2 matrix. Although this attempt alone failed, by conceptually expanding the failed equation we were able to obtain the final 4x4 matrix. First, we define the following: (0 is zero)

```
      (1  0)     (0 -1)
 Ed = (0 -1) D = (1  0)
```

D means different
(Ed is the deferent of the identity element)

We will also write the results first, but we set another matrix as

```
     (0 1)
 S = (1 0) (S means same)
```

and tried to see if these three would form a product group. This alone did not work, but we found that

(iEd) (D) (iS)

satisfy the conditions for a quaternion.

Below is the proof (i is the imaginary unit)... (*1)

```
 (1 0) (0-1)   (0 -1)
 (0-1) (1 0) = (-1 0)
----------------------------------
 Ed * D = -S Therefore
----------------------------------
(iEd) * D = -(iS)
 > equivalent to a quaternion

 (0-1) (0 1)   (-1 0)
 (1 0) (1 0) = (0  1)
----------------------------------
 D * S = -Ed Therefore
----------------------------------
 D * (iS) = -(iEd)
 > equivalent to a quaternion

 (0 1) (1 0)   (0-1)
 (1 0) (0-1) = (1 0)
---------------------------------
 S * Ed = + D Therefore
---------------------------------
 (iS) * (iEd) = - D
 >This is also equivalent to a
quaternion
```

<Matrix representation without using the imaginary unit i>

We were able to obtain a product group equivalent to a quaternion, but this was obtained by forcibly adding an existing imaginary unit as a borrowed entity, so this does not represent a quaternion in matrix representation. Here, we will use the property that

```
      (0 -1)
 i ≡ (1  0) = D.
```

The method of "expanding" a 2*2 matrix to a 4*4 matrix can be thought of as:

```
       (a b)
 A22 = (c d)

       (a*E b*E)
 A44 = (c*E d*E)
```

(E is the identity element for multiplication of a 2*2 square matrix)
Of course,

A22 ≡ A44 Here,
i*A22 ≡ D*A44,

and as D and E are both 2*2 square matrices, we can think of D as a coefficient of a 4*4 matrix
(Copilot described this as an expansion to a tensor. I don't know the details.)
Therefore, we consider

i*Ed22 ≡ D*Ed44 (*2)
i*S22 ≡ D*S44 (*3).

Therefore, by expanding the matrix, we obtain a 4 * 4 matrix.

*O: O is the 2 * 2 zero matrix.
*E is the 2 * 2 multiplication identity.

```
        (E  O)
 Ed44 = (O -E)

       (O -E)
 D44 = (E  O) (*4)

       (O E)
 S44 = (E O)
```

Consider this. Treating D as the imaginary unit in matrices as a coefficient, we obtain the combination equivalent to equation *1.

```
 (D  O) (O -E) (O D)
 (O -D) (E  O) (D O)
```

As a result of operational verification, we were able to confirm that multiplicationbehaves the same as quaternions.

(The proof is omitted because it is homologous to *1; the proof is at the end of the file.)

I mentioned *4 earlier, but it can be derived from the product of *2 and *3.

<Another combination>

Interestingly, we have found another set of matrices equivalent to quaternions. In the previous section, we extended and substituted the imaginary behavior matrix D in place of i, but this time we will consider the imaginary behavior in terms of a 4*4 skeleton. In other words, in the previous section, the extension was

iEd≡D*Ed44
iS ≡D*S44,

but we can now think of it as

iEd≡Ed*D44
iS ≡ S*D44.

This means that we cannot use D44 in the position of D in 2*2 (it will break down when we do multiplication). Here, (E and E44 are identity elements for multiplication)

D44 = E*D44

(an element of the combination from the previous section), so we can similarly reverse the positions of the coefficients and skeleton and think of it as D≡D*E44. When this is combined with the new extension 4*4 of iEd and iS, this also matches the product group as a quaternion.

<Product proof of Ed*D44 D*E44 S*D44>

```
Ed*D=-S Therefore

 (O -Ed)   (D O)   (O  S)      (O -S)
 (Ed  O) * (O D) = (-S O) = (-)(S  O)

 EdD44 * DE44 = −SD44

 D*S=-Ed Therefore

 (D O)   (O -S)   (O  Ed)      (O -Ed)
 (O D) * (S  O) = (-Ed O) = (-)(Ed  O)

 DE44 * SD44 = −EdD44

 S*Ed=+D Therefore

 (O -S)   (O -Ed)   (-D O)      (D O)
 (S  O) * (Ed  O) = (O -D) = (-)(O D)

 SD44 * EdD44 = -DE44
```

This concludes the illustration of the important concepts.

<Rigorous Verification>

I am not well-informed as to whether this is called a tensor operation, and I am not sure whether this alone will be enough to prove it, so as a check I will write down the verification in elementary expression of 4 x 4. Parentheses, zeros, and zero matrices have been omitted to avoid complicating things. The vertical and horizontal lines in the middle are guide lines.

<'Gold' i tendency from D22>

・Squared Check

```
    -1|              -1|           -1     |
 1    |           1    |               -1|
-------------  * -------------  =  -------------
      |     1          |     1           |-1
      |-1              |-1               |    -1

 (D  O)^2
 (O -D) = -E44

      |-1               |-1          -1    |
      |    -1           |    -1          -1|
-------------  *  -------------  =  -------------
 1    |            1    |                |-1
     1|                1|                |    -1

 (O -E)^2
 (E  O) = -E44

      |    -1           |    -1      -1    |
      | 1               | 1              -1|
-------------  *  -------------  =  -------------
    -1|               -1|                |-1
 1    |            1    |                |    -1

 (O D)^2
 (D O) = -E44
```

・Confirmation of cumulative group behavior

```
    -1|                 |-1             |     1            |    -1
 1    |                 |    -1         |-1                | 1
-------------  *  ------------- = ------------- = (-)-------------
      |     1      1    |              1|                -1|
      |-1              1|         -1    |              1   |

 (D  O)   (O -E)   (O -D)      (O D)
 (O -D) * (E  O) = (-D O) = (-)(D O)

      |-1                |    -1        1|               -1|
      |    -1            | 1       -1    |            1    |
-------------  *  ------------- = ------------- = (-)------------
 1    |               -1|               |    -1           |     1
     1|             1   |               | 1               |-1

 (O -E)   (O D)   (-D O)       (D  O)
 (E  O) * (D O) = (O  D) = (-) (O -D)

      |    -1         -1|               | 1                |-1
      | 1         1     |               |     1            |    -1
-------------  *  ------------- = -------------- = (-)------------
    -1|                 |     1   -1    |              1   |
 1    |                 |-1           -1|                 1|

 (O D)   (D  O)   (O  E)       (O -E)
 (D O) * (O -D) = (-E O) = (-) (E  O)

----------------------------------------
<'Plum' i tendency from D44>
```

・Squared Check

```
      |-1                |-1         -1     |
      |    1             |    1           -1|
------------  *  -------------  =  --------------
 1    |           1     |                 |-1
    -1|               -1|                 |    -1

 (O -Ed)^2
 (Ed  O) = -E44

    -1|               -1|           -1     |
 1    |            1    |                -1|
-------------  *  -------------  =  -------------
      |    -1           |    -1            |-1
      | 1               | 1                |    -1

 (D O)^2
 (O D) = -E44

      |    -1           |    -1     -1     |
      |-1               |-1              -1|
-------------  *  -------------  =  -------------
     1|                1|                  |-1
 1    |            1    |                  |    -1

 (O -S)^2
 (S  O) = -E44
```

・Confirmation of cumulative group behavior

```
      |-1             -1|               |     1            |    -1
      |     1      1    |               | 1                |-1
-------------  *  ------------- = ------------- = (-)-------------
 1    |                 |    -1       -1|                 1|
    -1|                 | 1       -1    |            1    |

 (O -Ed)   (D O)   (O  S)      (O -S)
 (Ed  O) * (O D) = (-S O) = (-) (S  O)

    -1|                 |    -1         | 1                |-1
 1    |                 |-1             |    -1            |     1
-------------  *  ------------- = ------------- = (-)-------------
      |    -1          1|         -1    |            1    |
      | 1          1    |              1|              -1|

 (D O)   (O -S)   (O  Ed)      (O -Ed)
 (O D) * (S  O) = (-Ed O) = (-) (Ed  O)

      |    -1            |-1            1|              -1|
      |-1                |     1   -1    |            1    |
-------------  *   ------------- = ------------- = (-)------------
     1|            1    |               |     1            |    -1
 1    |               -1|               |-1                | 1

 (O -S)   (O -Ed)   (-D O)      (D O)
 (S  O) * (Ed  O) = (O -D) = (-) (O D)
```

<What's the point of just using compound notation?>

As a joke, I tried giving nicknames to 4*4 matrices. This was an idea I came up with 30 years ago, so I have to chuckle at how poorly it felt at the time. I got two pairs, so I wanted contrasting names, and I was also a little skeptical about the Western leaning of general mathematical symbols, so I decided to use Chinese characters. 金(kim,Gold) and 李(Li,Plum) are common surnames in China and Korea. Both have auspicious meanings. One is a mineral, the other a lush tree. I came up with this pair by considering whether to use a compact 2*2 imaginary behavior matrix or one that uses the full 4*4 space, but it seems my old self ended up adopting this by stretching the idea of compact being a mineral and lush space being a tree. Let's summarize the definitions again here

----

```
                             (D   O)
 金Ed = GoldEd = D * Ed44 = (O - D)

                              (O -E)
 金(D) = Gold(D) = E * D44 = (E  O)

                            (O S)
 金S = GoldS   = D * S44 = (S O)
```

```
                            (O -Ed)
 李Ed = PlumEd = Ed * D44 = (Ed  O)

                             (D O)
 李(D) = Plum(D) = D * E44 = (O D)

                           (O -S)
 李S = PlumS   = S * D44 = (S  O)
```

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*If we strictly define here that

"Gold is something whose coefficient matrix is D"

and

"Plum is something whose skeleton matrix has the properties of D",

then Gold(D) and li(D) would be written in reverse. However, if these multiplication groups are closed and do not interfere, it would be difficult to understand if names written with different Chinese characters were mixed in with the groups, so I left it as it is, as it is also a convenient name. Alternatively, it might be better to write Gold(D*E44) li(E*D44), but if the only reason for the names is to make a distinction, this would also be a complicated format.

<Development - Commutative Groups>

According to my memory of trials 30 years ago, when these two groups of six matrices are multiplied and "outcrossed," nine new matrices are obtained, which respectively constitute three non-interfering multiplicative commutative groups. Explaining this would be too complicated, so I will omit it.
However, as a generated product, there is the following relationship between them. This is described below. The commutative group matrices obtained by combining the intersections of the horizontal Gold and the vertical Plum are the respective commutative group matrices. Here, I will simply write "Cmt"

```
      Gold Gold Gold
 Plum Cmt  Cmt  Cmt
 Plum Cmt  Cmt  Cmt
 Plum Cmt  Cmt  Cmt
```

If this were to be redesigned artistically, it would look like

```
 E    Gold Gold Gold E
 Plum Cmt  Cmt  Cmt  Plum
 Plum Cmt  Cmt  Cmt  Plum
 Plum Cmt  Cmt  Cmt  Plum
 E    Gold Gold Gold E
```

It looks exactly like the mandala of the Womb Realm. Buddhist auspicious numbers and Buddhist art reflect a sense of Indian mathematics.

:I have heard that group theory can also be seen in Islamic geometric designs (New Mathematics Study Method, Kodansha Bluebacks by Key Tohyama)

<Appendix: How i≡D was discovered>

Without finding this, the extension to quaternion behavior would not have been possible. It was discovered by chance.
Consider an arbitrary point on a circle of radius 1. Let the angle at that point from the reference line be theta. The matrix for rotational transformation on the xy plane is as follows (this can be proven using elementary geometry):

```
 cos(theta) -sin(theta)
 sin(theta)  cos(theta)
```

This can be transformed into the following using matrix addition:

```
            (1 0)            (0 -1)
 cos(theta)*(0 1)+sin(theta)*(1  0)

            : E              : D
```

This was important. I realized that this seemed to be homologous to de Moivre's formula, which indicates a point on a unit circle of the same radius 1 on the complex plane, rather than the xy plane, and since in

cos(theta)+i*sin(theta),

i^2 = -1, if we also square D, the concept equivalent to -1 in a matrix is the negative value of the integral identity, -E. Doing some simple calculations, I found that

```
       (0 -1)   (0 -1)   (-1 0)
 D^2 = (1  0) * (1  0) = (0 -1) = -E
```

I was pinched by the nose. It felt like discovery was a painful experience. (30 years ago) It is one proverb in Japan. 'fox pinched your nose' when someone feels sence of wonder. :)

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end of docment
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